$$\begin{cases} \text{MGF} \rightarrow \text{Laplace transform} & \mathbb{E}e^{tX} \\ \text{c.f.} \rightarrow \text{Fourier transform} & \mathbb{E}e^{itX} \end{cases}$$

$$\begin{cases} \text{real exponential} \rightarrow \text{monotone, explode/vanish} \\ \text{complex exponential} \rightarrow \text{periodic} \end{cases}$$

c.f. has 1-to-1 correspondence with dist from Levy's inversion formula (inverse Fourier)

good tool to deal with sum of independent r.v.

5.7.7: $X_1, \dots, X_n$ independent, $X_i \sim N(\mu_i, 1)$, $Y = X_1^2 + \dots + X_n^2$, calculate c.f. of Y.

Pf:

$$\phi_Y(t) = \prod_{j=1}^{n} \phi_{X_j^2}(t), \quad \text{and}$$

$$\phi_{X_j^2}(t) = \mathbb{E}\, e^{itX_j^2}$$

$$= \int_{-\infty}^{+\infty} e^{itx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_j)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu_j^2}{2}} \int_{-\infty}^{+\infty} e^{(it-\frac{1}{2})x^2 + \mu_j x} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu_j^2}{2}} \int_{-\infty}^{+\infty} e^{(it-\frac{1}{2})\left(x + \frac{\mu_j}{2it-1}\right)^2} e^{-\left(\frac{\mu_j}{2it-1}\right)^2 \cdot (it-\frac{1}{2})} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu_j^2\left(1 + \frac{1}{2it-1}\right)} \cdot \underbrace{\int_{\mathbb{R}} e^{(it-\frac{1}{2})\left(x+\frac{\mu_j}{2it-1}\right)^2} dx}_{\sqrt{\frac{\pi}{-(it-\frac{1}{2})}}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\mu_j^2}{2}\frac{2it}{2it-1}} \cdot \sqrt{\pi} \cdot \left(\frac{1}{2} - it\right)^{-\frac{1}{2}}$$

$$= (1-2it)^{-\frac{1}{2}} \cdot e^{\frac{it\mu_j^2}{1-2it}}$$

since

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

for $\forall a \in \mathbb{C}$, $\mathrm{Re}(a) < 0$.

$$\phi_Y(t) = (1-2it)^{-\frac{n}{2}} \cdot e^{\frac{it}{1-2it} \cdot \sum_{j=1}^{n} \mu_j^2}$$

Remark: $\int_{\mathbb{R}} e^{ax^2}\,dx \cdot \int_{\mathbb{R}} e^{ay^2}\,dy \overset{\text{polar}}{=} \int_0^{2\pi}\int_0^{+\infty} e^{ar^2}\cdot r\,dr\,d\theta$

$= \frac{2\pi}{a}$, taking square root on both sides proves the argument (or by analytic continuation)

5.7.10: X, Y cts r.v., show

$$\int_{-\infty}^{+\infty} \phi_X(y) f_Y(y) e^{-ity}\,dy = \int_{-\infty}^{+\infty} \phi_Y(x-t) f_X(x)\,dx$$

pf: $\text{RHS} = \int_{-\infty}^{+\infty} \mathbb{E}\, e^{i(x-t)Y} f_X(x)\,dx$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x-t)y} f_Y(y)\,dy\, f_X(x)\,dx$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} e^{-ity} f_Y(y) \underbrace{\int_{\mathbb{R}} e^{ixy} f_X(x)\,dx}_{\mathbb{E}\, e^{iyX} = \phi_X(y)}\,dy$$

$$= \int_{\mathbb{R}} \phi_X(y) f_Y(y) e^{-ity}\,dy.$$

set $t=0$, $\underbrace{\int_{\mathbb{R}} \phi_X(y) f_Y(y)\,dy}_{= \mathbb{E}\,\phi_X(Y)} = \underbrace{\int_{\mathbb{R}} \phi_Y(x) f_X(x)\,dx}_{= \mathbb{E}\,\phi_Y(X)}$ (Parseval relation) e.g. Triangular Dist, $f(x) = 1-|x|$, $x \in (-1,1)$

c.f. $\phi_X(t) = \int_{-1}^{1} (1-|x|)\, e^{itx}\, dx$

$$= \int_0^1 (1-x)e^{itx}\,dx + \int_{-1}^{0} (1+x)e^{itx}\,dx$$

$$= \frac{1}{it}\left(-1 + \frac{e^{it}-1}{it}\right) + \frac{1}{it}\left(1 - \frac{1-e^{-it}}{it}\right)$$

$$= \frac{2-2\cos t}{t^2}$$

Then for a RW $\{S_n\}$ on $\mathbb{R}^d$, μ_n as dist of S_n,

S_n departs from $S_0 = 0$ and want to investigate

prob of S_n going back to 0, $\forall \delta > 0$,

$$\mathbb{P}\left(\|S_n\| < \tfrac{1}{\delta}\right) = \int_{\|t\| < \frac{1}{\delta}} d\mu_n(t)$$

(if $|x| \le \frac{\pi}{3}$, $1-\cos x \ge \frac{x^2}{4}$)

$$\le \int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{4}{\delta^2 t_i^2}\left[1-\cos(\delta t_i)\right] d\mu_n(t)$$

$S_n \sim \mu_n$

$X = (X_1, \dots, X_d)$

$X_i \overset{\text{i.i.d.}}{\sim}$ Triangular

$$= 2^d \int_{\mathbb{R}^d} \underbrace{\prod_{i=1}^{d} \frac{2(1-\cos(\delta t_i))}{\delta^2 t_i^2}}_{\text{c.f.}}\ \underbrace{d\mu_n(t)}_{\text{dist}}$$

apply Parseval relation!

$$= 2^d \int_{(-\delta,\delta)^d} \underbrace{\prod_{i=1}^{d} \frac{\delta - |x_i|}{\delta^2}}_{\text{density}}\ \underbrace{\phi_{S_n}(x)}_{\text{c.f.}}\, dx$$

☆ Key step in building judging criterion for the recurrence of random walks!

5.8.b: $X_1, \dots, X_n$ be such that $\forall\, a_1, \dots, a_n \in \mathbb{R}$, $\sum_{i=1}^n a_i X_i$ is always Gaussian. Show that joint c.f. of X is $e^{it^T\mu - \frac{1}{2}t^T V t}$ for some $\mu \in \mathbb{R}^n$, $V \in \mathbb{R}^{n\times n}$. Show that X has multivariate Gaussian density if V is invertible.

Pf:

$$\phi_X(t) = \mathbb{E}e^{it^TX} = \mathbb{E}e^{i\underbrace{(t_1X_1 + \dots + t_nX_n)}_{\text{Gaussian, denote } Y_t}}$$

$$= e^{i\cdot\mu(t) - \frac{1}{2}\sigma^2(t)}$$

for some $\mu, \sigma : \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} \mathbb{E}Y_t = \mu(t) = \sum_{i=1}^n t_i\, \mathbb{E}X_i = t^T \cdot \underbrace{\mu}_{\text{mean vector of } X} \\ \mathrm{Var}\, Y_t = \sigma^2(t) = t^T \cdot \underbrace{V}_{\text{covariance matrix of } X} t \end{cases}$$

If V is invertible, since V is symmetric and semi-positive-definite, consider $Z = V^{-\frac{1}{2}}(X-\mu)$

$$\phi_Z(t) = e^{it^TV^{-\frac{1}{2}}(X-\mu)} = e^{-it^TV^{-\frac{1}{2}}\mu} \cdot \phi_X(V^{-\frac{1}{2}}t)$$

$$= e^{-it^TV^{-\frac{1}{2}}\mu} \cdot e^{i\cdot t^TV^{-\frac{1}{2}}\mu - \frac{1}{2}t^TV^{-\frac{1}{2}}VV^{-\frac{1}{2}}t} = e^{-\frac{1}{2}t^Tt}$$

So: $Z = V^{-\frac{1}{2}}(X-\mu) \sim N(0, I_n)$, which has density as the product of n Gaussian marginals since components are independent.

$$f_Z(z) = (2\pi)^{-\frac{n}{2}} e^{-\frac{z^T z}{2}}$$

Then, $X = V^{\frac{1}{2}} Z + \mu$ also admits a density

$$f_X(x) = f_Z\left(V^{-\frac{1}{2}}(x-\mu)\right) \cdot \det\left(\frac{\partial z}{\partial x}\right) \leftarrow \text{Jacobian matrix}$$

$$= (2\pi)^{-\frac{n}{2}} \cdot e^{-\frac{(x-\mu)^T V^{-1}(x-\mu)}{2}} \cdot \left[\det(V)\right]^{-\frac{1}{2}}$$

c.f. has natural connection with moments:

$$\phi_X'(t) = \mathbb{E}\left(iXe^{itX}\right), \text{ so } \phi_X'(0) = i \cdot \mathbb{E}X$$

if $X \in \mathcal{L}^1$

e.g.: If $\phi_X(t) = 1 + o(t^2)$ $(t \to 0)$, then $\phi_X \equiv 1$.

Pf: $\lim_{t\to 0} \frac{\phi_X(t)-1}{t^2} = 0$, so $\phi_X'(0) = \lim_{t\to 0} \frac{\phi_X(t)-1}{t} = 0$

which means $\mathbb{E}X = 0$, and

$$\phi_X''(0) = \lim_{t\to 0} \frac{\phi_X(t)+\phi_X(-t)-2\phi(0)}{t^2} = 0, \text{ so}$$

$X \in \mathcal{L}^2$ and $\mathbb{E}X^2 = 0 \Rightarrow \mathrm{Var}\, X = 0 \Rightarrow X = 0$ a.s.

This implies $\phi_X \equiv 1$, so c.f. cannot be "too flat" in the neighborhood of 0.